CCFU Proof 20

Parent Decomposition: $P = S_1 \oplus_W S_2$

Given. $P = \mathbb{R}^5$, $\operatorname{sig}(P) = (3,2)$ [Proof 5].
$S_1 \subset P$, $\dim S_1 = 4$, $\operatorname{sig}(S_1) = (3,1)$ [Proof 4].
$S_2 \subset P$, $\dim S_2 = 4$, $\operatorname{sig}(S_2) = (2,2)$ [Proofs 3, 16].

Step 1 — $\dim(S_1 \cap S_2) = 3$.

$$\dim(S_1 \cap S_2) \geq \dim S_1 + \dim S_2 - \dim P = 4 + 4 - 5 = 3.$$

If $\dim(S_1 \cap S_2) = 4$ then $S_1 = S_2$, but $\operatorname{sig}(S_1) = (3,1) \neq (2,2) = \operatorname{sig}(S_2)$. Therefore $\dim(S_1 \cap S_2) = 3$.

It follows that $\dim(S_1 + S_2) = 4 + 4 - 3 = 5 = \dim P$, so $S_1 + S_2 = P$. ■

Step 2 — W is nondegenerate.

S_1 is nondegenerate in P, so $S_1^\perp$ is 1-dimensional. $\operatorname{sig}(P) = (3,2)$ and $\operatorname{sig}(S_1) = (3,1)$, so $S_1^\perp$ has $\operatorname{sig}(0,1)$: it is a negative line.

S_2 is nondegenerate in P, so $S_2^\perp$ is 1-dimensional. $\operatorname{sig}(P) = (3,2)$ and $\operatorname{sig}(S_2) = (2,2)$, so $S_2^\perp$ has $\operatorname{sig}(1,0)$: it is a positive line.

Since a nonzero line cannot be both positive and negative, $S_1^\perp$ and $S_2^\perp$ are distinct. Let u span $S_1^\perp$ and v span $S_2^\perp$. Then u, v are linearly independent, $\langle u,u\rangle < 0$ and $\langle v,v\rangle > 0$, so the Gram determinant is

$$\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2 < 0.$$

Therefore $S_1^\perp + S_2^\perp$ is nondegenerate with $\operatorname{sig}(1,1)$.

$W = S_1 \cap S_2 = (S_1^\perp + S_2^\perp)^\perp$. The orthogonal complement of a nondegenerate subspace is nondegenerate. Therefore W is nondegenerate. ■

Step 3 — $\operatorname{sig}(W) = (2,1)$. W is 3-dimensional and nondegenerate.

In S_1 with $\operatorname{sig}(3,1)$: a 3d nondegenerate subspace has $\operatorname{sig} \in \{(3,0),\ (2,1)\}$.
In S_2 with $\operatorname{sig}(2,2)$: a 3d nondegenerate subspace has $\operatorname{sig} \in \{(2,1),\ (1,2)\}$.
Common: $\operatorname{sig}(W) = (2,1)$. ■

Step 4 — Orthogonal complements. Since W is nondegenerate, define $V_1 := S_1 \cap W^\perp$ and $V_2 := S_2 \cap W^\perp$. Then $S_1 = W \oplus^\perp V_1$ and $S_2 = W \oplus^\perp V_2$, with $\dim V_1 = \dim V_2 = 1$.

$$\operatorname{sig}(W) + \operatorname{sig}(V_1) = \operatorname{sig}(S_1): \quad (2,1) + \operatorname{sig}(V_1) = (3,1) \Rightarrow \operatorname{sig}(V_1) = (1,0).$$
$$\operatorname{sig}(W) + \operatorname{sig}(V_2) = \operatorname{sig}(S_2): \quad (2,1) + \operatorname{sig}(V_2) = (2,2) \Rightarrow \operatorname{sig}(V_2) = (0,1).$$

V_1 is positive. V_2 is negative. ■

Step 5 — Basis.

e_0, e_1 :	positive, in $W = S_1 \cap S_2$	$\operatorname{sig}(2,0)$
e_2 :	positive, in V_1 (S_1-only)	$\operatorname{sig}(1,0)$
e_3 :	negative, in $W = S_1 \cap S_2$	$\operatorname{sig}(0,1)$
e_4 :	negative, in V_2 (S_2-only)	$\operatorname{sig}(0,1)$

Choose unit vectors $e_2 \in V_1$ and $e_4 \in V_2$. Since e_0, e_1, e_3 are orthonormal in W and $V_1, V_2 \subset W^\perp$, the basis $(e_0, e_1, e_2, e_3, e_4)$ satisfies

$$\langle e_i, e_j\rangle = 0 \quad \text{whenever one of } i,j \in \{0,1,3\} \text{ and the other } \in \{2,4\}.$$

The only potentially nonzero off-diagonal entry is $\alpha := \langle e_2, e_4 \rangle$, which is determined by the fixed subspaces S_1, S_2.

Verification: $S_1 = \mathrm{span}(e_0, e_1, e_2, e_3)$, sig $= (3,1)$ ✓. $S_2 = \mathrm{span}(e_0, e_1, e_3, e_4)$, sig $= (2,2)$ ✓.

Remark. In an orthonormal basis of P (diagonalizing Q_5), the metric is $\mathrm{diag}(+1, +1, +1, -1, -1)$, but S_2 may not align with a coordinate subspace. The adapted basis above keeps S_1 and S_2 as coordinate subspaces at the cost of one off-diagonal entry α. ■

Conclusion. S_1 and S_2 are not chosen—they are the linear closure (Proof 4) and nonlinear closure (Proofs 3, 16) of C_2. Hence

$$W = S_1 \cap S_2$$

is fixed. The proof shows that W is 3-dimensional, nondegenerate, and has signature $(2, 1)$. The orthogonal complement lines

$$V_1 = S_1 \cap W^\perp, \qquad V_2 = S_2 \cap W^\perp$$

are also fixed by S_1, S_2, with $\mathrm{sig}(V_1) = (1, 0)$ and $\mathrm{sig}(V_2) = (0, 1)$. In an adapted basis their mutual inner product

$$\alpha = \langle e_2, e_4 \rangle$$

may appear as one off-diagonal metric entry; it is determined by the fixed pair (S_1, S_2) and is not an extra choice. For the downstream chain, however, no data from α is used: Proof 21 depends only on W and constructs

$$V_W = \mathbb{R}\tau \oplus W \oplus W^*.$$

Thus no additional parameter enters the higher alternating closure.

[Dependencies: Proofs 3, 4, 5, 16.]